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Generalized expressions for secondary vorticity using intrinsic co-ordinates

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Equations for the development of streamwise (or secondary) vorticity, for stationary and rotating systems, are derived directly from the vector equation for vorticity, using intrinsic co-ordinates. This approach places emphasis on the coupled equations for the vorticity components parallel and normal to the streamline. The equations derived are more general than those hitherto available, being valid for compressible, stratified and viscous flow. Some interpretation is given of the physical mechanism causing secondary flow.

1. Introduction

A streamwise component of vorticity is developed from the deflexion of a flow with velocity, or density gradients. Some examples of these secondary flows are the following.

(*a*) A sheared flow passing around an obstacle (e.g. the flow of water past a bridge pier (Hawthorne 1954)).

(*b*) A developed pipe or channel flow entering a bend (Hawthorne 1951; Marris 1963).

(*c*) A stratified fluid encountering a bend (Scorer & Wilson (1963) have studied the influence of this secondary vorticity in inducing secondary instability in gravity waves).

(*d*) The end-wall boundary layer passing through a cascade of turbine or compressor blades (this phenomenon has been extensively studied in view of its important engineering application in turbomachinery aerodynamics (Hawthorne 1966; Horlock & Lakshminarayana 1973; Lakshminarayana & Horlock 1963)).

(*e*) Curvature of a flow with a temperature gradient binormal to the curvature (e.g. flow into a turbine nozzle row from a combustion chamber (Loos 1956)).

(*f*) Flow in rotating passages, such as those in a turbine or compressor, where the entry velocity or density may be non-uniform (e.g. Smith 1957).

(*g*) Vortex motions induced in atmospheric and ocean currents by the earth's rotation (Marris 1966).

Squire & Winter (1951) first obtained an expression for the secondary vorticity developed by an incompressible shear flow passing through a bend. Hawthorne (1951) also studied the secondary vorticity generation quantitatively, deriving a general expression for the secondary vorticity developed in incompressible inviscid flow. He later (1954) used the (Helmholtz) vorticity equation to derive a particular form for the secondary vorticity in inviscid and incompressible flows. Other investigators (Marris 1963, 1966; Scorer & Wilson 1963; Loos 1956; Smith 1957) followed lines similar to that of Hawthorne (1951); all these analyses involve considerable vector manipulation.

The objective of this paper is first to state the vorticity equation in its most general vector form and then to use stationary intrinsic co-ordinates directly to derive expressions for the growth in secondary vorticity. This approach places the emphasis on coupled equations for two components of vorticity, streamwise and normal, both of which need to be solved. It also gives more general equations which can be simplified in special cases, e.g. (*a*) inviscid uniform-density flow, (*b*) inviscid incompressible stratified flow, (*c*) inviscid compressible flow, (*d*) viscous incompressible flow, (*e*) inviscid incompressible flow with body forces. Corresponding equations for a rotating co-ordinate system are then given.

2. Secondary vorticity in a stationary system

The equation for the vorticity $\boldsymbol{\omega}$ in compressible viscous flow is

$$\underset{(1)}{(\mathbf{V}.\nabla)\boldsymbol{\omega}} = \underset{(2)}{(\boldsymbol{\omega}.\nabla)\mathbf{V}} - \underset{(3)}{\boldsymbol{\omega}(\nabla.\mathbf{V})} - \underset{(4)}{\nabla\times\left(\frac{\nabla p}{\rho}\right)} - \underset{(5)}{\nabla\times\left[\frac{\mu}{\rho}\nabla\times\boldsymbol{\omega} - \frac{4}{3}\frac{\mu}{\rho}\nabla(\nabla.\mathbf{V})\right]} \quad (1)$$

if the bulk viscosity is taken as zero. Here $\mathbf{V}$ is the velocity, $\boldsymbol{\omega} = \nabla\times\mathbf{V}$, ρ is the density, p is the static pressure and μ is the viscosity. The incompressible form of this equation can be found in most textbooks (e.g. Batchelor 1967). Terms 3 and 4 arise because of the effect of compressibility or density stratification alone.

Term 5 may be written as

$$\underset{(5)}{-\nabla\times\left[\frac{\mu}{\rho}\nabla\times\boldsymbol{\omega} - \frac{4}{3}\frac{\mu}{\rho}\nabla(\nabla.\mathbf{V})\right]} = \underset{(6)}{\frac{\mu}{\rho}\nabla^2\boldsymbol{\omega}} - \underset{(7)}{\nabla\left(\frac{\mu}{\rho}\right)\times\nabla\times\boldsymbol{\omega}} + \underset{(8)}{\tfrac{4}{3}\nabla\left(\frac{\mu}{\rho}\right)\times\nabla(\nabla.\mathbf{V})}. \quad (2)$$

Term 6 is due to viscosity alone, and we shall use it in the analysis of the secondary vorticity developed in the flow of a viscous incompressible fluid (§2.3, case (*d*) below). Terms 7 and 8 involve combinations of the effects of gradients of viscosity and density; if the fluid is compressible then it is rarely justifiable to assume μ is constant, and gradients of μ should be considered. However, we shall not study the flow of a viscous compressible fluid here.

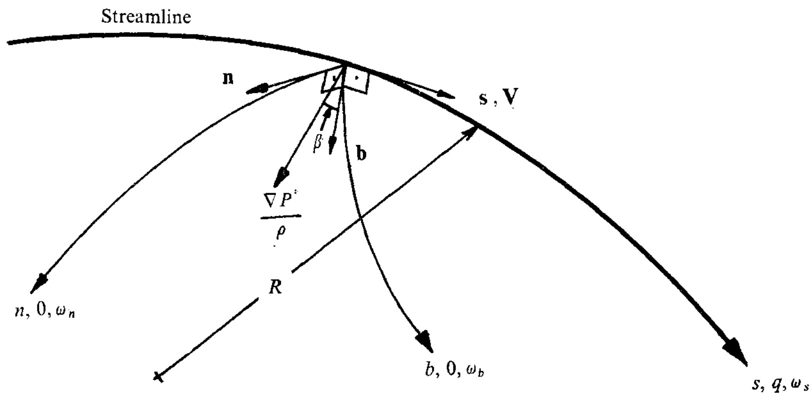


FIGURE 1. Notation used for stationary system.

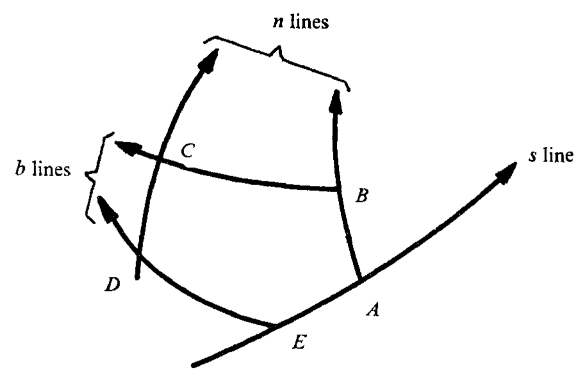


FIGURE 2. s, n and b lines when $\omega_s \neq 0$ (Bjørgum 1951).

We now derive a generalized expression for the vorticity components in an intrinsic co-ordinate system (figure 1).

After Bjørgum (1951), we first define the unit vector along the streamline. The vector velocity $\mathbf{V} = \mathbf{s}q$, where q is the magnitude of the velocity. The unit principal normal vector $\mathbf{n}$ is then defined from $\mathbf{n}/R = \mathbf{s}.\nabla\mathbf{s} = \partial\mathbf{s}/\partial s$, where R is the principal radius of curvature. The unit binormal vector $\mathbf{b}$ is $\mathbf{s}\times\mathbf{n}$, so that $(\mathbf{s}, \mathbf{n}, \mathbf{b})$ is a right-handed set of unit vectors. Our natural co-ordinate system is based on the vector lines $\mathbf{s}$, $\mathbf{n}$, and $\mathbf{b}$ but it should be noted that only in special cases do there exist orthogonal surfaces containing these lines. (Bjørgum shows that a complex lamellar flow, with the streamwise vorticity non-zero, is one such special case. But in general, with this vorticity non-zero, if we move along a circuit $ABCDE$ made up of constant-n and constant-b lines (figure 2) we do not return to the point A.)

We do not, therefore, follow the usual type of analysis involving the Lamé coefficients in orthogonal curvilinear co-ordinate systems, but consider various relationships arising from the manipulation and differentiation of the unit vectors.

Some of these relationships valid for intrinsic co-ordinates, given by Bjørgum (1951), are

$$\mathbf{V}.\nabla = q\frac{\partial}{\partial s}, \quad \mathbf{n}.\nabla = \frac{\partial}{\partial n}, \quad \mathbf{b}.\nabla = \frac{\partial}{\partial b}, \tag{3}$$

$$\nabla = \mathbf{s}\frac{\partial}{\partial s}+\mathbf{n}\frac{\partial}{\partial n}+\mathbf{b}\frac{\partial}{\partial b}, \tag{4}$$

$$\nabla.\mathbf{V} = \frac{\partial q}{\partial s}+q\nabla.\mathbf{s} = -\mathbf{V}.\frac{\nabla\rho}{\rho}, \tag{5}$$

$$\begin{aligned}\nabla\times\mathbf{V} &= \mathbf{s}\xi q+\mathbf{n}\frac{\partial q}{\partial b}+\mathbf{b}\left(\frac{q}{R}-\frac{\partial q}{\partial n}\right)\\ &= \mathbf{s}\omega_s+\mathbf{n}\omega_n+\mathbf{b}\omega_b.\end{aligned} \tag{6}$$

The secondary vorticity component $\omega_s = \xi q$ requires special discussion. The quantity ξ is independent of vector magnitude and is defined by $\xi = \mathbf{s}.\nabla\times\mathbf{s}$. Bjørgum calls this 'the torsion of neighbouring vector lines'.

The differential coefficients of the unit vectors $\mathbf{s}$, $\mathbf{n}$ and $\mathbf{b}$ are

$$\left.\begin{aligned}&\frac{\partial\mathbf{s}}{\partial s} = \frac{\mathbf{n}}{R}, \quad \frac{\partial\mathbf{s}}{\partial n} = \frac{\mathbf{n}}{a_n}\frac{\partial a_n}{\partial s}, \quad \frac{\partial\mathbf{s}}{\partial b} = \frac{\mathbf{b}}{a_b}\frac{\partial a_b}{\partial s},\\ &\frac{\partial\mathbf{n}}{\partial s} = \frac{\mathbf{b}}{\tau}-\frac{\mathbf{s}}{R}, \quad \frac{\partial\mathbf{b}}{\partial s} = -\frac{\mathbf{n}}{\tau},\end{aligned}\right\} \tag{7}$$

where a_n is the distance in the n direction between neighbouring streamlines, a_b is the distance in the b direction between neighbouring streamlines and τ is the radius of torsion of the streamline (see Hildebrand 1962).

It should be noted that the direction of $\mathbf{n}$ taken here is towards the centre of curvature, whereas Hawthorne and Loos have taken the outward direction for $\mathbf{n}$.

2.1. *Streamwise component of vorticity*

The equation for the streamwise component of (1) can be derived by taking the dot product of the equation with $\mathbf{s}$. We consider first the inviscid terms in the equation.

First term:

$$\mathbf{s}.(\mathbf{V}.\nabla)\boldsymbol{\omega} = q\mathbf{s}.\frac{\partial}{\partial s}(\boldsymbol{\omega}) = q\frac{\partial}{\partial s}(\mathbf{s}.\boldsymbol{\omega})-q\boldsymbol{\omega}.\frac{\partial\mathbf{s}}{\partial s} = q\frac{\partial\omega_s}{\partial s}-q\frac{\omega_n}{R}. \tag{8}$$

Second term:

$$\mathbf{s}.(\boldsymbol{\omega}.\nabla)\mathbf{V} = \omega_s\frac{\partial q}{\partial s}+\omega_n\frac{\partial q}{\partial n}+\omega_b\frac{\partial q}{\partial b} = \omega_s\frac{\partial q}{\partial s}+\frac{q\omega_n}{R}. \tag{9}$$

Third term:

$$\mathbf{s}.\boldsymbol{\omega}(\nabla.\mathbf{V}) = -\frac{1}{\rho}\mathbf{s}.\boldsymbol{\omega}(\mathbf{V}.\nabla\rho) = -\frac{\omega_s}{\rho}q\frac{\partial\rho}{\partial s}. \tag{10}$$

Fourth term:

$$\mathbf{s}.\nabla\times\left(\frac{\nabla p}{\rho}\right) = \frac{1}{\rho^2}\left[\frac{\partial p}{\partial n}\frac{\partial\rho}{\partial b}-\frac{\partial\rho}{\partial n}\frac{\partial p}{\partial b}\right]. \tag{11}$$

The viscous terms become extremely complex in the (s, n, b) co-ordinates so for the time being we leave them in vector form.

When (8)–(11) are substituted in (1), and the terms rearranged, the equation for the streamwise vorticity becomes

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{\rho q}\right) = \frac{2\omega_n}{\rho q R} - \frac{1}{q^2\rho^3}\left[\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial b} - \frac{\partial \rho}{\partial n}\frac{\partial p}{\partial b}\right] + \frac{\mu}{\rho^2 q^2}\mathbf{s}.\nabla^2\boldsymbol{\omega}$$
$$-\frac{1}{\rho q^2}\mathbf{s}.\nabla\left(\frac{\mu}{\rho}\right)\times(\nabla\times\boldsymbol{\omega}) + \frac{4}{3}\frac{\mathbf{s}}{\rho q^2}.\nabla\left(\frac{\mu}{\rho}\right)\times\nabla(\nabla.\mathbf{V}). \qquad (12)$$

This is an exact equation, valid for fluids with viscosity μ, but with bulk viscosity zero. The appearance of the factor 2 in the second term arises from equal contributions from the substantial derivative $d\boldsymbol{\omega}/dt$ and the term $(\boldsymbol{\omega}.\nabla)\mathbf{V}$ respectively. Examination of the inviscid terms in (12) shows how secondary vorticity is developed (a) when there is a normal component of vorticity (ω_n) in a flow of radius of curvature R; (b) when density and pressure gradients exist in the b, n surface in mutually perpendicular directions (since $\nabla(\rho^{-1})$ and ∇p are normals respectively to surfaces of constant density and constant pressure, the vector $\nabla(\rho^{-1})\times\nabla p$ is tangential to the curve of intersection of these surfaces). Furthermore, even in the absence of these effects an existing secondary vorticity will change (owing to compressibility and velocity changes) through the term $\partial(\omega_s/\rho q)/\partial s$ owing to vortex stretching.

2.2. *Normal vorticity component*

Similarly the normal component of the vorticity can be derived from the dot product of (1) with $\mathbf{n}$. Again we consider the inviscid terms in detail.

First term:

$$\mathbf{n}.(\mathbf{V}.\nabla)\boldsymbol{\omega} = q\mathbf{n}.\frac{\partial\boldsymbol{\omega}}{\partial s} = q\frac{\partial}{\partial s}(\boldsymbol{\omega}.\mathbf{n}) - q\boldsymbol{\omega}.\frac{\partial\mathbf{n}}{\partial s}$$
$$= q\frac{\partial\omega_n}{\partial s} - \frac{q\omega_b}{\tau} + \frac{q\omega_s}{R} \qquad (13)$$

since from Frenet's formula (Hildebrand 1962)

$$\frac{d\mathbf{n}}{ds} = \frac{\mathbf{b}}{\tau} - \frac{\mathbf{s}}{R},$$

where τ is the radius of torsion.

Second term:

$$\mathbf{n}.(\boldsymbol{\omega}.\nabla)\mathbf{V} = \mathbf{n}.\left[\omega_s\frac{\partial}{\partial s}(q\mathbf{s}) + \omega_n\frac{\partial}{\partial n}(q\mathbf{s}) + \omega_b\frac{\partial}{\partial b}(q\mathbf{s})\right]$$
$$= \frac{\omega_s q}{R} + \frac{\omega_n q}{a_n}\left(\frac{\partial a_n}{\partial s}\right)$$
$$= \frac{\omega_s q}{R} + \omega_n q\left(-\frac{1}{\rho q}\frac{\partial}{\partial s}(\rho q) - \frac{1}{a_b}\frac{\partial a_b}{\partial s}\right). \qquad (14)$$

Third term:

$$\mathbf{n}.\boldsymbol{\omega}(\nabla.\mathbf{V}) = -\omega_n\left(\frac{\mathbf{V}.\nabla\rho}{\rho}\right) = \frac{-\omega_n q}{\rho}\frac{\partial\rho}{\partial s}. \qquad (15)$$

Fourth term:

$$\mathbf{n}.\nabla \times \left(\frac{\nabla p}{\rho}\right) = \frac{1}{\rho^2}\left[\frac{\partial p}{\partial b}\frac{\partial \rho}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial \rho}{\partial b}\right]. \tag{16}$$

The viscous terms are again left in vector form because of their complexity. Substituting (13)–(16) in (1), we obtain after some rearrangement

$$\frac{\partial}{\partial s}(\omega_n q) = \frac{q\omega_b}{\tau} - \frac{\omega_n q}{a_b}\frac{\partial a_b}{\partial s} + \frac{1}{\rho^2}\left[\frac{\partial p}{\partial s}\frac{\partial \rho}{\partial b} - \frac{\partial p}{\partial b}\frac{\partial \rho}{\partial s}\right] + \mathbf{n}.\frac{\mu}{\rho}\nabla^2\boldsymbol{\omega}$$
$$-\mathbf{n}.\nabla(\mu/\rho)\times\nabla\times\boldsymbol{\omega} + \tfrac{4}{3}\mathbf{n}.(\nabla(\mu/\rho)\times\nabla(\nabla.\mathbf{V})) \tag{17}$$

or

$$\rho q\frac{\partial}{\partial s}\left(\frac{\omega_n}{\rho}\right) = \frac{q\omega_b}{\tau} + \frac{\omega_n q}{a_n}\frac{\partial a_n}{\partial s} + \frac{1}{\rho^2}\left[\frac{\partial p}{\partial s}\frac{\partial \rho}{\partial b} - \frac{\partial p}{\partial b}\frac{\partial \rho}{\partial s}\right] + \text{viscous terms}. \tag{18}$$

Thus the normal vorticity changes continuously, and (12) and (18) are coupled. Marris (1963, 1966) has provided the incompressible form of the equation for streamwise vorticity, but this by itself is not sufficient to calculate ω_s. For a complete solution (12) and (18) should be solved simultaneously; viscosity is usually important only in (18).

2.3. *Special cases*

We consider first inviscid flows.

(*a*) *Inviscid uniform-density flow.* If the flow is inviscid and of uniform density, then (12) and (17) become

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q}\right) = \frac{2\omega_n}{qR}, \tag{19}$$

$$\frac{\partial}{\partial s}(\omega_n q) = \frac{q\omega_b}{\tau} - \frac{\omega_n q}{a_b}\frac{\partial a_b}{\partial s}. \tag{20}$$

The first of these equations is the most familiar equation for the development of streamwise vorticity in incompressible inviscid flow. Since the vortex lines and the streamlines lie in surfaces of constant stagnation pressure P

$$\mathbf{b}.\nabla P/\rho = \mathbf{b}.(\mathbf{V}\times\boldsymbol{\omega}) = \boldsymbol{\omega}.(\mathbf{b}\times\mathbf{V})$$
$$= (\boldsymbol{\omega}.\mathbf{n})q = q\omega_n = q(\partial q/\partial b). \tag{21}$$

Substituting (21) in (19) and integrating between an upstream station 1 and a downstream station 2 gives

$$\left(\frac{\omega_s}{q}\right)_2 - \left(\frac{\omega_s}{q}\right)_1 = 2\int_1^2 \frac{|\nabla P/\rho|\cos\beta\, ds}{q^2 R}, \tag{22}$$

where β is the angle between the direction of ∇P (the principal normal to the Bernoulli surfaces) and the binormal to the streamlines (figure 1). This is Hawthorne's (1951) expression for the secondary vorticity.

An alternative approximate form is given by assuming that the Bernoulli planes remain flat with $\tau = \infty$, and $\partial\mathbf{s}/\partial b = (\mathbf{b}/a_b)(\partial a_b/\partial s) = 0$. We may then write $q = q_p u$, where $q_p = q_p(s)$ is a potential flow velocity with upstream value of unity, and $u = u_1(b)$. Equation (20) then gives

$$\omega_n q_p = \omega_{n1} \quad \text{(constant)}. \tag{23}$$

Substituting (23) in (19) and integrating yields

$$\left(\frac{\omega_s}{q_p}\right)_2 - \left(\frac{\omega_s}{q_p}\right)_1 = 2\frac{du_1}{db}\int_1^2 \frac{ds}{q_p^2 R}, \tag{24}$$

which is Hawthorne's (1954) expression for secondary flow about struts and airfoils.

If q_p remains constant through the duct and $ds = r\,d\epsilon$, where ϵ is the turning angle,

$$\omega_{s2} - \omega_{s1} = 2\epsilon(du_1/db), \tag{25}$$

which is Squire & Winter's (1951) expression for secondary vorticity in a duct.

(*b*) *Incompressible inviscid stratified flow.* A stratified flow is defined as one in which the density does not change in the streamwise direction ($\partial\rho/\partial s = 0$), the separate terms in the continuity equation being zero:

$$\mathbf{V}.\nabla\rho = 0 = \rho\nabla.\mathbf{V}. \tag{26}$$

However, density gradients $\partial\rho/\partial n$ and $\partial\rho/\partial b$ may exist in a b, n surface, so that the terms in the square brackets in (12) and (18) must be retained.

But, since the flow is inviscid,

$$-\frac{\nabla p}{\rho} = -\mathbf{s}\frac{1}{\rho}\frac{\partial p}{\partial s} - \mathbf{n}\frac{1}{\rho}\frac{\partial p}{\partial n} - \mathbf{b}\frac{1}{\rho}\frac{\partial p}{\partial b} = (\mathbf{V}.\nabla)\mathbf{V} = \mathbf{s}q\frac{\partial q}{\partial s} + q^2\frac{\partial \mathbf{s}}{\partial s}$$
$$= \mathbf{s}q(\partial q/\partial s) + \mathbf{n}(q^2/R), \tag{27}$$

so that

$$\frac{\partial p}{\partial b} = 0, \quad \frac{\partial p}{\partial n} = -\frac{\rho q^2}{R}. \tag{28}$$

Equation (12) then becomes

$$\frac{\partial}{\partial s}(\omega_s/q) = \frac{2\omega_n}{qR} + \frac{1}{\rho R}\frac{\partial\rho}{\partial b} \tag{29}$$

$$= \frac{1}{R}\frac{\partial}{\partial b}\ln(\rho q^2), \tag{30}$$

the expression derived by Scorer & Wilson (1963) and Marris (1964) (see also Drazin (1961) and Hawthorne & Martin (1955), who give expressions for the secondary vorticity developed in this case). It should be noted that secondary vorticity develops in a stratified flow with a density gradient $\partial\rho/\partial b$ and principal radius of curvature R, even if $\omega_n = 0$. A density gradient $\partial\rho/\partial n$ may exist but does not affect the development of ω_s.

(*c*) *Compressible inviscid flow.* If the flow is inviscid but compressible

$$(\rho = \rho(s, n, b))$$

then the relations (27) and (28) are still valid and (12) is

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{\rho q}\right) = \frac{2\omega_n}{\rho q R} + \frac{1}{\rho^2 R}\frac{\partial\rho}{\partial b}. \tag{31}$$

However, if the fluid is barotropic, $p = f(\rho)$, then the term in the square brackets in (12) is identically zero (not only is $\partial p/\partial b$ zero, but also $\partial\rho/\partial b = f'^{-1}\partial p/\partial b = 0$). Then

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{\rho q}\right) = \frac{2\omega_n}{\rho q R}. \tag{32}$$

This equation would for example be valid for a homentropic flow of a perfect gas in which p/ρ^γ = constant throughout the fluid.

An alternative form to (31) (for inviscid flow of a compressible fluid, not necessarily a perfect gas, with entropy gradients) may be obtained by eliminating the vorticity component ω_n from that equation. Taking the dot product of $\mathbf{b}$ with Crocco's equation gives

$$\mathbf{b}.(\nabla h_0 - T\nabla S) = |\nabla h_0|\cos\beta_1 - T|\nabla S|\cos\beta_2 = q\omega_n. \tag{33}$$

β_1 and β_2 are the angles between the normals to the surfaces of constant stagnation enthalpy h_0 and entropy S and the binormal direction. The term $\rho^{-2}R^{-1}\,\partial\rho/\partial b$ in (31) arises from the term $\nabla \times \nabla p/\rho$ in the original vorticity equation and may be written alternatively in terms of temperature T and entropy S:

$$-\frac{1}{\rho q^2}\mathbf{s}.\nabla\times\frac{(\nabla p)}{\rho} = -\frac{\mathbf{V}}{\rho q^3}.\nabla\times(\nabla h - T\nabla S) = \frac{\mathbf{V}}{\rho q^3}.(\nabla T\times\nabla S).$$

Hence an alternative expression for the secondary vorticity growth in compressible inviscid flow is

$$\left(\frac{\omega_s}{\rho q}\right)_2 - \left(\frac{\omega_s}{\rho q}\right)_1 = 2\int_0^s \frac{(|\nabla h_0|\cos\beta_1 - T|\nabla S|\cos\beta_2)\,ds}{\rho q^2 R} + \int_0^s \frac{\mathbf{V}.(\nabla T\times\nabla S)\,ds}{\rho q^3}. \tag{34}$$

This expression is equivalent to that given by Loos (1956), although he does not distinguish between β_1 and β_2 and he expresses $\nabla T\times\nabla S$ in terms of stagnation temperature and $\boldsymbol{\omega}$. (His equation is inconvenient to use in view of the appearance of $\boldsymbol{\omega}$, which is unknown, on the right-hand side of the equation.) It should be noted here that compressibility really only introduces *additional* secondary vorticity when ∇T and ∇S are not in the same direction or when the pressure and density gradients are normal to each other in the b, n surface (equation 12).

For a perfect gas the secondary vorticity can also be expressed in terms of stagnation pressure P and density. This can be illustrated by expressing (31) in terms of static temperature. As $p/\rho T$ is now constant, the density gradient term is $\rho^{-1}\,\partial\rho/\partial b = -T^{-1}\partial T/\partial b$, since $\partial p/\partial b = 0$, and

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{\rho q}\right) = \frac{2\omega_n}{\rho q R} - \frac{1}{\rho R T}\frac{\partial T}{\partial b}. \tag{35}$$

Again for the perfect gas, the Crocco equation (33) can be written as (Hawthorne 1955*a*)

$$\mathbf{V}\times\boldsymbol{\omega} = C_p\left(1-\frac{T}{T_0}\right)\nabla T_0 + \frac{T}{T_0}\frac{1}{\rho_0}\nabla P.$$

The dot product of this equation with $\mathbf{b}$ provides

$$q\omega_n = C_p\left(1-\frac{T}{T_0}\right)\frac{\partial T_0}{\partial b} + \frac{T}{T_0}\frac{1}{\rho_0}\frac{\partial P}{\partial b}.$$

Furthermore

$$\frac{\partial T}{\partial b} = \frac{\partial T_0}{\partial b} - \frac{q\omega_n}{C_p}.$$

Using these equations, ω_n and T can be eliminated from (35) to give

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{\rho q}\right) = \frac{2}{\rho\rho_0 q^2 R}\frac{\partial P}{\partial b}. \tag{36}$$

A similar expression has been derived by Hawthorne (1955*a*), by another route. Although (36) is applicable for a perfect gas only, it is probably the most useful form of the equations for growth of secondary vorticity.

(*d*) *Incompressible viscous flow.* For this case the governing equations (12) and (17) become

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q}\right) = \frac{2\omega_n}{qR}+\frac{\mu \mathbf{s}.\nabla^2\boldsymbol{\omega}}{\rho q^2} \tag{37}$$

and
$$\frac{\partial}{\partial s}(\omega_n q) = \frac{q\omega_b}{\tau}-\frac{q\omega_n}{a_b}\frac{\partial a_b}{\partial s}+\frac{\mu \mathbf{n}.\nabla^2\boldsymbol{\omega}}{\rho},$$

or
$$q\frac{\partial \omega_n}{\partial s} = \frac{q\omega_b}{\tau}+\frac{\omega_n q}{a_n}\frac{\partial a_n}{\partial s}+\frac{\mu \mathbf{n}.\nabla^2\boldsymbol{\omega}}{\rho}. \tag{38}$$

Marris has given (37), but did not provide (38).

Louis (1956) has obtained an approximate solution to these equations, by assuming that viscous effects in a slightly deflected flow with a given entry shear are the same as the (known) viscous effects in an undeflected flow with the same entry shear.

Following Louis, the viscous flow about a two-dimensional body is considered as a superposition of two flows by writing $q = q_p u + u'$, where $q_p(s)$ is the potential flow velocity of the deflected flow with an upstream value of unity, $u(s, b)$ is the known viscous solution for velocity in the absence of deflexion of the flow, and u' is a small perturbation. Let $\omega_{n0} = \partial u(s,b)/\partial b$ be the normal velocity in the absence of flow deflexion. Then in this two-dimensional primary flow

$$\omega_b = 0 = \partial a_b/\partial s,$$

and (38) reduces to
$$\frac{\partial}{\partial s}(\omega_{n0}u) = \frac{\mu\nabla^2\omega_{n0}}{\rho}. \tag{39}$$

When there is flow deflexion, Louis assumes that the spanwise flow can be neglected (i.e. $\partial a_b/\partial s$ is very small and τ is very large), so that (38) becomes

$$\frac{\partial}{\partial s}[(\omega_{n0}+\omega_n')(q_p u+u')] = \frac{\mu}{\rho}\nabla^2(\omega_{n0}+\omega_n'), \tag{40}$$

where ω_n' is the perturbation in normal vorticity. From the difference between (39) and (40)

$$\partial[\omega_n' q_p u+(q_p-1)\,\omega_{n0}u+\omega_{n0}u']/\partial s \simeq 0, \tag{41}$$

neglecting the product term $\omega_n' u'$ and $(\mu/\rho)\nabla^2\omega_n'$. Louis argued that if the initial shear is weak and the deflexion is small then the third term on the left-hand side is of second order. Then approximately

$$\partial(\omega_n' q_p u)/\partial s = -\omega_{n0}u(\partial q_p/\partial s) \tag{42}$$

since $(q_p - 1)\,\partial(\omega_{n0} u)/\partial s$ is also of second order. Hence

$$\omega_n' \simeq -\frac{1}{q_p u}\int_0^s \frac{\partial u}{\partial b} u \frac{\partial q_p}{\partial s} ds \tag{43}$$

and

$$\omega_n = \frac{\partial u}{\partial b} - \frac{1}{q_p u}\int_0^s \frac{\partial u}{\partial b} u \frac{\partial q_p}{\partial s} ds. \tag{44}$$

The direct effect of viscosity on the streamwise equation is neglected, although the indirect effect through changing ω_n is allowed for. Thus

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q_p u}\right) \simeq \frac{1}{u}\frac{\partial}{\partial s}\left(\frac{\omega_s}{q_p}\right) = \frac{2\omega_n}{q_p u R}, \tag{45}$$

if it is again assumed that $\partial u/\partial s$ is small, but that $\partial q_p/\partial s$ must be retained. Hence, substituting (44) in (45),

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q_p}\right) = \frac{2(\partial u/\partial b)}{q_p R} - \frac{2}{q_p^2 R u}\int_0^s u \frac{\partial u}{\partial b}\frac{\partial q_p}{\partial s} ds \tag{46}$$

or

$$\frac{\omega_s}{q_p} = \int_0^s \frac{2}{q_p}\frac{\partial u}{\partial b}\frac{ds}{R} - \int_0^s \frac{2}{q_p^2 R u}\left[\int_0^s u \frac{\partial u}{\partial b}\frac{\partial q_p}{\partial s} ds\right] ds. \tag{47}$$

Equations (44) and (47) are identical to those of Louis (1956). Essentially this approach comes down to solving the equation

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q}\right) = \frac{2\omega_n}{qR} \tag{48}$$

using values of q and ω_n obtained from $q = q_p u$, where q_p and u are known from potential flow and a two-dimensional viscous flow respectively.

(*e*) *The effect of body forces in inviscid incompressible flow.* Some understanding of the generation of the total streamwise vorticity in flow past a body or bodies is obtained by supposing that there are distributed body forces $\mathbf{F}/\rho\ (s, n, b)$ per unit volume in the flow, simulating the action of wings or blades. The secondary vorticity equation for incompressible inviscid flow then becomes

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\omega_s}{q}\right) &= \frac{2\omega_n}{qR} + \frac{\mathbf{s}}{\rho q^2}.(\nabla \times \mathbf{F}) \\ &= \frac{2\omega_n}{qR} + \frac{1}{\rho q^2}\left[\frac{\partial F_b}{\partial n} - \frac{\partial F_n}{\partial b}\right].\end{aligned} \tag{49}$$

If Hawthorne's (1967) small shear, large deflexion assumption is made, then the Bernoulli surfaces do not distort. For such a flow through a cascade or around a wing, with Bernoulli planes lying in surfaces of constant b, and body forces existing in the n direction,

$$\begin{aligned}F_n = \mathbf{n}.\mathbf{F} &= -\rho\mathbf{n}.(\mathbf{s}q \times \mathbf{b}\gamma) \\ &= \rho q \gamma,\end{aligned} \tag{50}$$

where γ is the distributed bound vorticity replacing the blades (figure 3), and

$$\mathbf{b}.\mathbf{F} = F_b = 0.$$

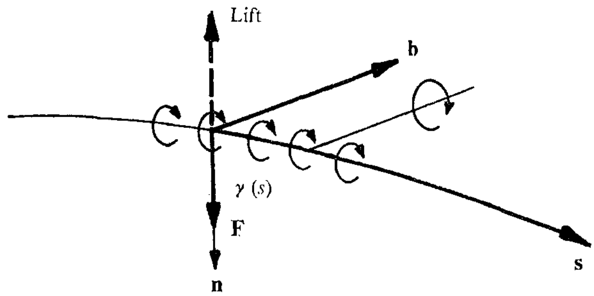


FIGURE 3. Distributed bound vorticity replacing the blade or wing.

Hence

$$\frac{\partial}{\partial s}\left(\frac{\omega_s}{q}\right) = \frac{2\omega_n}{qR} - \frac{1}{q^2}\frac{\partial}{\partial b}(q\gamma) \tag{51}$$

and on integrating,

$$\left(\frac{\omega_s}{q}\right)_2 - \left(\frac{\omega_s}{q}\right)_1 = 2\int_0^s \frac{\omega_n}{qR}\,ds - \int_0^s \frac{1}{q}\frac{d\gamma}{db}\,ds - \int_0^s \frac{\gamma}{q^2}\frac{\partial q}{\partial b}\,ds. \tag{52}$$

Approximately, if q is nearly constant along a streamline,

$$\omega_{s2} - \omega_{s1} = 2\int_0^s \frac{\omega_n}{R}\,ds - \frac{d\Gamma}{db} - \frac{\Gamma}{q}\frac{dq}{db}, \tag{53}$$

where $\Gamma = \int_0^s \gamma\, ds$, the total bound circulation distributed along the streamline.

Additional streamwise vorticity arises, associated with this bound circulation. Hawthorne (1955*b*) has designated these additional components as 'trailing shed vorticity' (the gradient of Γ, usually associated with shed vorticity) and 'trailing filament vorticity' ($\Gamma q^{-1}\, dq/db$) respectively.

In flow through a cascade the first component of vorticity is contained within the passage between the blades, and the second and third components within the blade wakes. As the blade spacing is reduced, until the blades may be represented by distributed body forces **F**, so the streamwise components of vorticity come together until all three may be thought of as lying along the unit vector **s** of the streamline, as described by (53).

3. Secondary vorticity in a rotating system

The secondary vorticity generation in a rotating frame is of particular importance in geophysical fluid mechanics and turbomachinery aerodynamics.

The equation of motion with reference to axes rotating at constant angular velocity $\mathbf{\Omega}$ is (Greenspan 1968, p. 5)

$$\boldsymbol{\zeta} \times \mathbf{W} + 2\mathbf{\Omega} \times \mathbf{W} = -\nabla' p/\rho - \nabla'\left[\tfrac{1}{2}W^2 - \tfrac{1}{2}(\mathbf{\Omega}\times\mathbf{r}).(\mathbf{\Omega}\times\mathbf{r})\right] - \frac{\mu}{\rho}\nabla'\times\nabla'\times\mathbf{W} + \frac{4}{3}\frac{\mu}{\rho}\nabla'(\nabla'.\mathbf{W}), \tag{54}$$

where the prime denotes differentiation with respect to the rotating frame, **W** is the relative velocity and $\boldsymbol{\zeta}$ is the relative vorticity ($\boldsymbol{\zeta} = \nabla' \times \mathbf{W} = \boldsymbol{\omega} - 2\mathbf{\Omega}$).

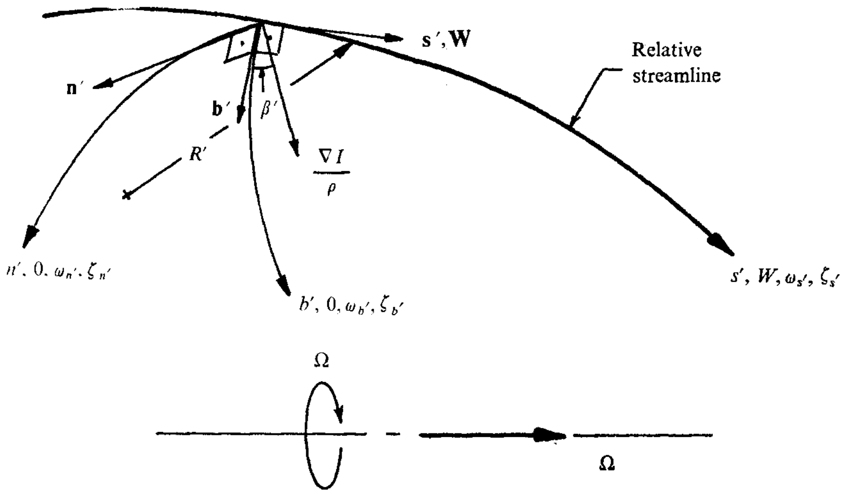


FIGURE 4. Notation used for rotating system.

The curl of (54) gives

$$(\mathbf{W}.\nabla')\boldsymbol{\zeta} = (\boldsymbol{\zeta}.\nabla')\mathbf{W} - 2\nabla'\times(\boldsymbol{\Omega}\times\mathbf{W}) - \nabla'\times(\rho^{-1}\nabla'p) - \boldsymbol{\zeta}(\nabla'.\mathbf{W})$$
$$\quad (1) \qquad\qquad (2) \qquad\qquad (3) \qquad\qquad (4) \qquad\qquad (5)$$
$$+\mu\rho^{-1}\nabla'^2\boldsymbol{\zeta} - \nabla'(\mu\rho^{-1})\times\nabla'\boldsymbol{\zeta} + \tfrac{4}{3}\nabla'(\mu\rho^{-1})\times\nabla'(\nabla'.\mathbf{W}). \quad (55)$$
$$(6) \qquad\qquad (7) \qquad\qquad (8)$$

To derive a generalized secondary vorticity expression, the intrinsic co-ordinates (shown in figure 4) will be employed. $\mathbf{s}', \mathbf{n}'$ and $\mathbf{b}'$ are now unit vectors along the relative streamline, principal normal and binormal directions respectively, so that $\mathbf{W} = \mathbf{s}'W$, $\boldsymbol{\zeta} = \mathbf{s}'\zeta_{s'} + \mathbf{n}'\zeta_{n'} + \mathbf{b}'\zeta_{b'}$. The relationships (3)–(7) derived for this (s, n, b) co-ordinate system are also valid for this (s', n', b') system.

3.1. *Streamwise component of relative vorticity* ($\zeta_{s'}$)

The procedure used in deriving the streamwise component of vorticity from (55) is similar to that used in the stationary systems. The streamwise components of terms 1, 2, and 4–8 are similar to those derived in §2.1, and only the third term, $-2\nabla'\times(\boldsymbol{\Omega}\times\mathbf{W})$, requires special consideration.

Since $\boldsymbol{\Omega}$ is a constant vector and $\mathbf{W} = \mathbf{s}'W$,

$$\mathbf{s}'.2\nabla'\times(\boldsymbol{\Omega}\times\mathbf{W}) = 2\mathbf{s}'.\boldsymbol{\Omega}(\nabla'.\mathbf{W}) - 2\mathbf{s}'.(\boldsymbol{\Omega}.\nabla')\mathbf{W}$$
$$= 2\mathbf{s}'.\boldsymbol{\Omega}(\nabla'.\mathbf{W}) - 2(\boldsymbol{\Omega}.\nabla')W - 2W\mathbf{s}'.(\boldsymbol{\Omega}.\nabla')\mathbf{s}' \quad (56)$$

but

$$2\mathbf{s}'.\boldsymbol{\Omega}(\nabla'.\mathbf{W}) = 2\mathbf{s}'.\boldsymbol{\Omega}\left(-\frac{\mathbf{W}.\nabla'\rho}{\rho}\right) = -2\mathbf{s}'.\boldsymbol{\Omega}\left(\frac{W}{\rho}\frac{\partial\rho}{\partial s}\right),$$

$$(\boldsymbol{\Omega}.\nabla')W = \boldsymbol{\Omega}.(\nabla'W),$$

$$W\mathbf{s}'.(\boldsymbol{\Omega}.\nabla')\mathbf{s}' = W\mathbf{s}'.\left(\Omega_{s'}\frac{\partial\mathbf{s}'}{\partial s'} + \Omega_{n'}\frac{\partial\mathbf{s}'}{\partial n'} + \Omega_{b'}\frac{\partial\mathbf{s}'}{\partial b'}\right) = 0,$$

so that

$$\mathbf{s}'.2\nabla'\times(\boldsymbol{\Omega}\times\mathbf{W}) = -2\Omega_{s'}\frac{W}{\rho}\frac{\partial\rho}{\partial s'} - 2\boldsymbol{\Omega}.(\nabla'W). \quad (57)$$

Hence, for the rotating case

$$\underset{(1)}{\frac{\partial}{\partial s'}\left(\frac{\zeta_{s'}}{\rho W}\right)} = \underset{(2)}{\frac{2\zeta_{n'}}{\rho W R'}} - \underset{(3)}{\frac{1}{\rho^3 W^2}\left[\frac{\partial \rho}{\partial b'}\frac{\partial p}{\partial n'} - \frac{\partial \rho}{\partial n'}\frac{\partial p}{\partial b'}\right]} + \underset{(4)}{2\Omega_{s'}\frac{W}{\rho}\frac{\partial \rho}{\partial s'}}$$

$$+ \underset{(5)}{\frac{2\mathbf{\Omega}.(\nabla' W)}{\rho W^2}} + \underset{(6)}{\frac{\mu \mathbf{s}'}{\rho^2 W^2}.\nabla'^2\boldsymbol{\zeta}} - \underset{(7)}{\frac{1}{\rho W^2}\mathbf{s}'.\nabla'\left(\frac{\mu}{\rho}\right)\times\nabla'\times\boldsymbol{\zeta}}$$

$$+ \underset{(8)}{\frac{4}{3}\frac{1}{\rho W^2}\mathbf{s}'.\nabla'\left(\frac{\mu}{\rho}\right)\times\nabla'(\nabla'.\mathbf{W})}, \tag{58}$$

where R' is the radius of curvature of the relative streamline.

The means by which the streamwise component of vorticity is produced in a relative flow are similar to those in a stationary system, namely through deflexion of the relative flow with a normal vorticity component and the existence of density and pressure gradients normal to each other in the b', n' surface (note that $\partial p/\partial b'$ is not now zero in inviscid flow) but additional terms 4 and 5 exist because of the effect of rotation. It is interesting to note that even if the relative vorticity $\boldsymbol{\zeta}$ is initially zero, contraction or curvature of the streamlines ($\nabla' W \neq 0$) subsequently produces secondary vorticity even in incompressible flow. For this reason geophysical flows are almost always rotational and the interaction of the relative vorticity and so called planetary vorticity $2\mathbf{\Omega}$ is a central feature of geophysical fluid dynamics (Pedlosky 1972).

Marris (1966) suggests that the term $2\mathbf{\Omega}.\nabla' W/\rho W^2$ dominates in comparison to the term $2\zeta_{n'}/\rho W R'$ in (58) (i.e. that 'secondary vorticity generation from the rotation must greatly exceed any effect from the streamline curvature') but this is not always the case. Consider for example the inviscid incompressible flow through a rotating row of compressor blades where the inlet absolute vorticity $\boldsymbol{\omega}$ is zero and assume that the relative streamlines lie on a cylindrical surface, whose axis lies along $\mathbf{\Omega}$ so that $\mathbf{\Omega}.\mathbf{b}' = 0$. The change of W with s' is neglected ($\partial W/\partial s' = 0$) and $\partial W/\partial n' = W/R'$, since there is no relative or absolute vorticity in the b' direction. Then since $\boldsymbol{\zeta} = -2\mathbf{\Omega}$

$$\frac{2\zeta_{n'}}{WR'} = 2\,\frac{-2\mathbf{\Omega}.\mathbf{n}'}{WR'} = -\frac{4\Omega_{n'}}{WR'}$$

and

$$\frac{2\mathbf{\Omega}.(\nabla' W)}{W^2} = \frac{2\mathbf{\Omega}.\mathbf{n}'}{W^2}\frac{\partial W}{\partial n'} = \frac{2\Omega_{n'}}{WR'}.$$

Hence, the secondary vorticity generation due to the fifth term in (58) is of the same order as that due to the second term.

Note that the *absolute* vorticity resolved along the relative streamline does not change in this case, since

$$\frac{\partial}{\partial s'}\left(\frac{\omega_{s'} - 2\mathbf{\Omega}.\mathbf{s}'}{W}\right) = -\frac{2\mathbf{\Omega}.\mathbf{n}'}{WR'}$$

or

$$\frac{\partial}{\partial s'}\left(\frac{\omega_{s'}}{W}\right) = -\frac{2\mathbf{\Omega}.\mathbf{n}'}{WR'} + \frac{2\mathbf{\Omega}}{W}.\frac{\partial \mathbf{s}'}{\partial s'} = 0.$$

3.2. *Absolute vorticity* $\boldsymbol{\omega}$ *along the relative streamline*

Equation (58) can be recast into a form that provides an expression for the development of *absolute* vorticity along the relative streamline ($\omega_{s'}$). This form is useful in many practical applications, e.g., turbomachinery flows.

The fifth term in (58) can be expressed as

$$\frac{2\boldsymbol{\Omega}.\nabla' W}{\rho W^2} = \frac{2\boldsymbol{\Omega}}{\rho W^2}.\left(\mathbf{s}'\frac{\partial W}{\partial s'}+\mathbf{n}'\frac{\partial W}{\partial n'}+\mathbf{b}'\frac{\partial W}{\partial b'}\right). \tag{59}$$

Now,

$$\zeta_{s'} = \omega_{s'} - 2\boldsymbol{\Omega}.\mathbf{s}', \tag{60}$$

$$\zeta_{n'} = \omega_{n'} - 2\boldsymbol{\Omega}.\mathbf{n}' = \partial W/\partial b, \tag{61}$$

$$\zeta_{b'} = \omega_{b'} - 2\boldsymbol{\Omega}.\mathbf{b}' = -\partial W/\partial n' + W/R', \tag{62}$$

where $\omega_{s'}$, $\omega_{n'}$ and $\omega_{b'}$ are the components of absolute vorticity along the s', n' and b' directions respectively.

Using (61) and (62), equation (59) can be expressed as

$$\begin{aligned}\frac{2\boldsymbol{\Omega}.(\nabla' W)}{\rho W^2} &= \frac{2\boldsymbol{\Omega}.\mathbf{s}'}{\rho W^2}\frac{\partial W}{\partial s'} + \frac{2\boldsymbol{\Omega}}{\rho W^2}.\left(\mathbf{n}'\frac{\partial W}{\partial n'} - \mathbf{b}'(2\boldsymbol{\Omega}.\mathbf{n}')\right) + \frac{2\boldsymbol{\Omega}.\mathbf{b}'\omega_{n'}}{\rho W^2} \\ &= \frac{2\boldsymbol{\Omega}.\mathbf{s}'}{\rho W^2}\frac{\partial W}{\partial s'} + \frac{2\boldsymbol{\Omega}.\mathbf{n}'}{\rho W^2}\left(\frac{W}{R'} - \omega'_b\right) + \frac{2\Omega_{b'}\omega_{n'}}{\rho W^2}.\end{aligned} \tag{63}$$

Substituting (63) in (58), noting that

$$\partial(\boldsymbol{\Omega}.\mathbf{s}')/\partial s = \boldsymbol{\Omega}.\mathbf{n}'/R'$$

and rearranging the terms, yields

$$\begin{aligned}\frac{\partial}{\partial s'}\left(\frac{\zeta_{s'}+2\Omega_{s'}}{\rho W}\right) &= \frac{2}{\rho W R'}(\zeta_{n'}+2\Omega_{n'}) + \frac{2\Omega_{b'}\omega_{n'}}{\rho W^2} - \frac{2\Omega_{n'}\omega_{b'}}{\rho W^2} \\ &\quad - \frac{1}{\rho^3 W^2}\left[\frac{\partial\rho}{\partial b'}\frac{\partial p}{\partial n'} - \frac{\partial\rho}{\partial n'}\frac{\partial p}{\partial b'}\right] + \text{viscous terms},\end{aligned} \tag{64}$$

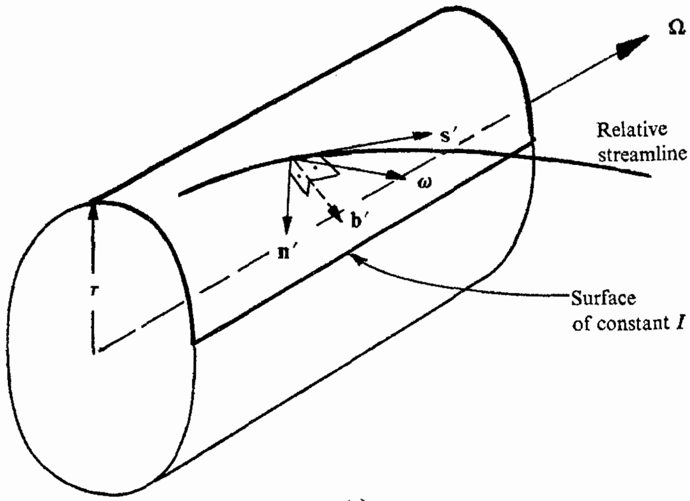
or

$$\begin{aligned}\underset{(1)}{\frac{\partial}{\partial s'}\left(\frac{\omega_{s'}}{\rho W}\right)} &= \underset{(2)}{\frac{2\omega_{n'}}{\rho W R'}} + \underset{(3)}{\frac{(2\Omega_{b'}\omega_{n'} - 2\Omega_{n'}\omega_{b'})}{\rho W^2}} \\ &\quad \underset{(4)}{- \frac{1}{\rho^3 W^2}\left[\frac{\partial\rho}{\partial b'}\frac{\partial p}{\partial n'} - \frac{\partial\rho}{\partial n'}\frac{\partial p}{\partial b'}\right]} + \text{viscous terms}.\end{aligned} \tag{65}$$

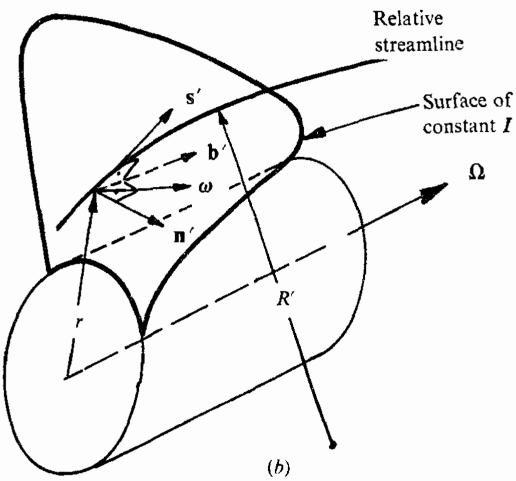
Comparison of this equation with the inviscid form of (12) shows the effect of rotation. The addition terms due to rotation can be expressed as

$$\begin{aligned}\frac{2\boldsymbol{\Omega}.\mathbf{b}'\omega_{n'} - 2\boldsymbol{\Omega}.\mathbf{n}'\omega_{b'}}{\rho W^2} &= \frac{2}{\rho W^2}[(\boldsymbol{\Omega}.\mathbf{b}')(\boldsymbol{\omega}.\mathbf{n}') - (\boldsymbol{\Omega}.\mathbf{n}')(\boldsymbol{\omega}.\mathbf{b}')] \\ &= \frac{2}{\rho W^2}(\boldsymbol{\Omega}\times\boldsymbol{\omega}).(\mathbf{b}'\times\mathbf{n}') = -\frac{2\boldsymbol{\Omega}\times\boldsymbol{\omega}}{\rho W^2}.\mathbf{s}'.\end{aligned}$$

Thus additional secondary vorticity (absolute) is generated when $\boldsymbol{\Omega}\times\boldsymbol{\omega}$ has



(a)



(b)

FIGURE 5. Surfaces of constant I. (a) Cylindrical. (b) Radial.

a component in the relative streamwise direction. Rotation has no effect when the absolute vorticity vector lies in the s, n plane and the rotation $\mathbf{\Omega}$ has no component in the binormal direction.

For example, consider the case where the planes of constant I† are cylindrical, with $\boldsymbol{\omega}$ in these surfaces and $2\mathbf{\Omega}$ lying parallel to the axis of the cylinders, as at entry to an axial compressor rotor, receiving a non-uniform velocity profile. Then $\mathbf{\Omega} \times \boldsymbol{\omega}$ is in the b direction (figure 5a) so rotation contributes no additional secondary vorticity. The generation of $\omega_{s'}$ is mainly due to $\omega_{n'}$, and/or the density gradients as in a stationary system.

On the other hand, if the surfaces of constant I are radial planes, as in a centrifugal turbomachine (figure 5b), the rotation-induced secondary vorticity may be

† $\nabla' I/\rho = \mathbf{W} \times \boldsymbol{\omega}$, see below, § 3.5.

quite appreciable. For radially outward non-uniform flow (with $\omega_{n'} \neq 0$) and radial blading, $\mathbf{\Omega}.\mathbf{n}' = 0$, $\mathbf{\Omega}.\mathbf{b}' = \Omega$, $R' = \infty$ and $\omega_{b'} = 0$. Secondary vorticity is produced only by the effect of rotation. For swept-back or swept-forward blading, secondary vorticity is produced by rotation and by curvature of the relative flow. The ratio of the third term to the second term (rotation-induced secondary vorticity to curvature-induced secondary vorticity) in (65) is

$$\frac{\mathbf{\Omega}.\mathbf{b}'R'}{W} = \frac{\Omega R'}{W} = \left(\frac{\Omega r}{W}\right)\frac{R'}{r} = \frac{U}{W}\frac{R'}{r},$$

where r is the distance from the axis of rotation and $U = \Omega r$ is the blade speed. U/W is large at the tip of a centrigual compressor (of order 4:1) and R'/r is certainly not less than unity, so that rotation-induced secondary vorticity will still dominate.

3.3. *The normal component of relative vorticity* ($\zeta_{n'}$)

The procedure for deriving the normal component of (55) is similar to that used in deriving (13)–(18). The only additional term to be evaluated is

$$\mathbf{n}'.2\nabla'\times\mathbf{\Omega}\times\mathbf{W} = 2\mathbf{n}'.\mathbf{\Omega}(\nabla'.\mathbf{W}) - 2\mathbf{n}'.(\mathbf{\Omega}.\nabla')\mathbf{W} - 2\Omega_{n'}\frac{W}{\rho}\frac{\partial\rho}{\partial s'} - 2W\mathbf{n}'.(\mathbf{\Omega}.\nabla')\mathbf{s}'. \quad (66)$$

But
$$\mathbf{n}'.(\mathbf{\Omega}.\nabla')\mathbf{s}' = \mathbf{n}'.\left(\Omega_{s'}\frac{\partial\mathbf{s}'}{\partial s'} + \Omega_{n'}\frac{\partial\mathbf{s}'}{\partial n'} + \Omega_{b'}\frac{\partial\mathbf{s}'}{\partial b'}\right)$$
$$= \frac{\Omega_{s'}}{R'} + \frac{\Omega_{n'}}{a_{n'}}\frac{\partial a_{n'}}{\partial s'} = \frac{\Omega_{s'}}{R'} - \frac{\Omega_{n'}}{a_{b'}}\frac{\partial a_{b'}}{\partial b'} - \frac{\Omega_{n'}}{\rho W}\frac{\partial(\rho W)}{\partial s'}.$$

Hence, (66) can be expressed as

$$\mathbf{n}'.2\nabla'\times\mathbf{\Omega}\times\mathbf{W} = -\frac{2\Omega_{s'}W}{R'} + \frac{2W\Omega_{n'}}{a_{b'}}\frac{\partial a_{b'}}{\partial b'} + \frac{2\Omega_{n'}\partial W}{\partial s'}.$$

Hence, for the rotating case, the equation corresponding to (17) is

$$\underset{(1)}{\frac{\partial}{\partial s'}(\zeta_{n'}W)} = \underset{(2)}{\frac{W\zeta_{b'}}{\tau}} - \underset{(3)}{\frac{W}{a_{b'}}\frac{\partial a_{b'}}{\partial s'}}\,\underset{(4)}{(\zeta_{n'} + 2\Omega_{n'})} - \underset{(5)}{2\Omega_{n'}\frac{\partial W}{\partial s'}} + \underset{(6)}{\frac{2\Omega_{s'}W}{R'}} + \underset{(7)}{\frac{1}{\rho^2}\left[\frac{\partial p}{\partial s'}\frac{\partial\rho}{\partial b'} - \frac{\partial p}{\partial b'}\frac{\partial\rho}{\partial s'}\right]} + \text{viscous terms}. \quad (67)$$

The mechanism by which the normal relative vorticity component changes is the same as in the stationary system but the rotation brings additional changes through the terms 4, 5 and 6 in (67). The additional effects due to rotation are similar to those generating the secondary vorticity (i.e. the rotation has the effect of producing normal vorticity in a flow with curvature or velocity gradients, even if the relative vorticity is initially zero).

3.4. *Normal component of absolute vorticity* ($\omega_{n'}$)

As was mentioned earlier, it is useful in many practical situations to express the normal vorticity equation in terms of absolute vorticity change. Noting that

$$\frac{\partial}{\partial s'}(2\mathbf{\Omega}.\mathbf{n}') = 2\mathbf{\Omega}.\frac{\partial \mathbf{n}'}{\partial s'} = 2\mathbf{\Omega}.\left(\frac{\mathbf{b}'}{\tau}-\frac{\mathbf{s}'}{R'}\right)$$

and using (61) and (62), equation (67) can be expressed as

$$\frac{\partial}{\partial s'}(\omega_{n'}W) = \frac{W}{\tau}\omega_{b'} - \frac{W}{a_b}(\omega_{n'})\frac{\partial a_{b'}}{\partial s'} - 2\Omega_{n'}\frac{\partial W}{\partial s'} + \frac{1}{\rho^2}\left[\frac{\partial p}{\partial s'}\frac{\partial \rho}{\partial b'} - \frac{\partial p}{\partial b'}\frac{\partial \rho}{\partial s'}\right] + \text{viscous terms}. \tag{68}$$

3.5. *Special cases*

In stationary co-ordinates, simplified expressions for secondary vorticity were given for several special cases. The corresponding cases for rotating co-ordinates can be derived, from the general equations, but we give here only two examples: (*a*) inviscid uniform-density flow; (*b*) incompressible inviscid stratified fluid.

(*a*) *Inviscid uniform-density flow.* In this case, the expressions involving the *absolute* vorticity will be used to show that the authors' generalized expressions reduce to those of Smith (1957).

For incompressible flow, (65) and (68) reduce to

$$\frac{\partial}{\partial s}\left(\frac{\omega_{s'}}{W}\right) = \frac{2\omega_{n'}}{WR'} + \frac{2\Omega_{b'}\omega_{n'}}{W^2} - \frac{2\Omega_{n'}\omega_{b'}}{W^2} = \frac{2\omega_{n'}}{WR'} - \frac{2\mathbf{s}'.(\mathbf{\Omega}\times\boldsymbol{\omega})}{W^2}, \tag{69}$$

$$\frac{\partial}{\partial s'}(\omega_{n'}W) = \frac{W\omega_{b'}}{\tau} - \frac{W\omega_{n'}}{a_{b'}}\frac{\partial a_{b'}}{\partial s'} - 2\Omega_{n'}\frac{\partial W}{\partial s'}. \tag{70}$$

These equations, more general than Smith's, are valid exactly for incompressible inviscid flow. The first can be expressed exactly in a form given by Smith (1957).

Smith notes that the vortex lines (absolute) lie in the (relative) Bernoulli planes:

$$\nabla' I/\rho = \mathbf{W}\times\boldsymbol{\omega},$$

where $$I = p + \tfrac{1}{2}\rho W^2 - \tfrac{1}{2}\rho(\Omega r)^2.$$

Thus, $$\mathbf{b}'.\nabla' I/\rho = \mathbf{b}'.(\mathbf{W}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}.\mathbf{n}')W = \omega_{n'}W,$$

i.e. $$|\nabla' I/\rho|\cos\beta' = W\omega_{n'}, \tag{71}$$

where β' is the angle between the binormal direction and the normal to the Bernoulli planes. Further,

$$\mathbf{\Omega}.(\mathbf{W}\times\boldsymbol{\omega}) = \mathbf{W}.(\boldsymbol{\omega}\times\mathbf{\Omega}) = -\mathbf{s}'.(\mathbf{\Omega}\times\boldsymbol{\omega})W = \mathbf{\Omega}.\nabla' I/\rho,$$

i.e. $$\Omega|\nabla' I/\rho W|\cos\delta = -\mathbf{s}'.(\mathbf{\Omega}\times\boldsymbol{\omega}), \tag{72}$$

where δ is the angle between $\mathbf{\Omega}$ and $\nabla' I/\rho$. Hence (69) becomes

$$\frac{\partial}{\partial s}\left(\frac{\omega_{s'}}{W}\right) = \frac{2}{W^2R'}\left|\frac{\nabla' I}{\rho}\right|\cos\beta' + \frac{2\Omega}{W^3}\left|\frac{\nabla' I}{\rho}\right|\cos\delta. \tag{73}$$

Consider the non-uniform flow through an axial compressor rotor in which the relative streamlines and absolute vortex lines lie on cylindrical surfaces ($\omega_{b'} = 0$) and the vector $\mathbf{\Omega}$ is parallel to the axis of these cylindrical surfaces ($\mathbf{\Omega}.\mathbf{b}' = 0$). Then if the flow is incompressible and inviscid and there is no variation of W along the streamlines, (70) gives $\omega_{n'}$ = constant and equation (69) simplifies to

$$(\omega_{s'})_2 - (\omega_{s'})_1 = 2\,\omega_{n'} \int_0^{s'} \frac{ds'}{R'} = 2\omega_{n'}\,\epsilon', \tag{74}$$

where ϵ' is the turning angle of the relative streamline. This is the equivalent of Squire & Winter's expression (equation 25) for a rotating case.

(*b*) *Incompressible inviscid stratified flow.* The study of secondary flow generation in a stratified fluid in a rotating frame is useful in understanding the vortex motion in atmospheric and ocean currents.

The equation of motion for inviscid stratified flow in a rotating frame of reference is after Marris (1966), for a point on the earth's surface

$$-\nabla p/\rho = \mathbf{W}.\nabla'\mathbf{W} + 2\mathbf{\Omega}\times\mathbf{W} - \mathbf{g}. \tag{75}$$

The gravitational acceleration $\mathbf{g}$ is retained here since it plays a significant role in geophysical fluid mechanics.

The dot product with $\mathbf{n}$ of (75) provides

$$-\frac{1}{\rho}\frac{\partial p}{\partial n'} = \frac{W^2}{R'} + 2\mathbf{n}.(\mathbf{\Omega}\times\mathbf{W}) - \mathbf{n}.\mathbf{g}. \tag{76}$$

Substituting for $\partial p/\partial n'$ (equation 76) in (58) and rearranging the terms, the secondary vorticity expression for incompressible inviscid flow stratified in the b' direction can be shown to be

$$\frac{\partial}{\partial s'}\left(\frac{\zeta_{s'}}{W}\right) = \underset{(1)}{\frac{1}{R'}\frac{\partial}{\partial b'}[\ln \rho W^2]} + \underset{(2)}{\frac{\partial \rho}{\partial b'}\frac{\mathbf{n}}{\rho W^2}}.\underset{(3)}{[2(\mathbf{\Omega}\times\mathbf{W}) - \mathbf{g}]} + \underset{(4)}{\frac{2\mathbf{\Omega}.\nabla' W}{W^2}}. \tag{77}$$

This expression was originally derived by Marris (1966) using first the kinematics of the vorticity. Marris expresses the second term in (77) in a form which can be shown to be the same as the authors':

$$\begin{aligned}\frac{2\mathbf{s}'.(\mathbf{\Omega}\times\mathbf{W})\times\nabla'\rho}{\rho W^2} &= \frac{2(\mathbf{n}'\times\mathbf{b}').(\mathbf{\Omega}\times\mathbf{W})\times\nabla'\rho}{\rho W^2} \\ &= (2/\rho W^2)\,[(\mathbf{n}'.\mathbf{\Omega}\times\mathbf{W})\,(\mathbf{b}'.\nabla'\rho) - (\mathbf{n}'.\nabla'\rho)\,(\mathbf{b}'.\mathbf{\Omega}\times\mathbf{W}) \\ &= \frac{2\mathbf{n}'.(\mathbf{\Omega}\times\mathbf{W})}{\rho W^2}\frac{\partial\rho}{\partial b'},\end{aligned}$$

since $\mathbf{n}.\nabla\rho = \partial\rho/\partial n' = 0$ in incompressible stratified flow.

It may be noted that equation (77) for the growth of relative streamwise vorticity in stratified flow in a rotating co-ordinate system is less general than equation (30) for the stationary co-ordinate system, in which it was not necessary to assume stratification in the b direction only.

This paper is dedicated to Sir William Hawthorne, who laid the groundwork of secondary flow theory, which has now reached the status of a 'classical' area of fluid mechanics comparable to potential flow theory.

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